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INTEGRO-DIFFERENTIAL EQUATION OF THE DYNAMIC CONTACT PROBLEM OF VISCOELASTICITY

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We consider the problem of impact of a perfectly rigid flat-ended stamp on a large viscoelastic plate, and study the case when a spring with a linear characteristic is inserted between the impacting stamp and the plate. The present paper represents a generalization of the results previously obtained in [1].

1. If we assume that the deflection $w(t)$ of a perfectly elastic plate [1] is connected with the force $P(t)$ acting on it by the following impulse relation

$$w(t) = \beta_0 \int_0^t P(t)\,dt, \qquad \beta_0^2 = \frac{3(1-\nu_0^2)}{16 E_0 \gamma h^4} \tag{1.1}$$

then the corresponding relation for the case when the elastic lag is present, will be obtained when the constant β_0 is replaced by the operator

$$\beta_t = \beta_0 (1 + H^*) \tag{1.2}$$

Here H^* is a positive integral operator with the lag kernel $H(t-\tau) > 0$, and we have

$$H^*\zeta = \int_0^t H(t-\tau)\,\zeta(\tau)\,d\tau \tag{1.3}$$

where ν_0 is a Poisson's ratio, E_0 is the modulus of elasticity, h is thickness, γ is the density of the plate material and $\zeta(t)$ is a function of time.

Thus in the present problem we shall derive the equation defining the force $P(t)$ to be determined, using the following relation as the starting point

$$w(t) = \beta_0 y(t) + \beta_0 \int_0^t H(t-\tau)\,y(\tau)\,d\tau, \quad y(t) = \int_0^t P(\xi)\,d\xi \tag{1.4}$$

The total displacement of the stamp is equal to $s + w$, where s denotes the compression of the spring, therefore the momentum equation for the stamp

$$mv(t) = mv_0 - y(t)$$

where $v(t)$ is the velocity of the stamp of mass m and v_0 is the initial velocity of the stamp, yields, with the relation $P = ks$, where k is the spring rigidity coefficient taken into account,

$$\frac{1}{k}\frac{d^2P}{dt^2} + \frac{d^2w}{dt^2} + \frac{P(t)}{m} = 0 \tag{1.5}$$

The initial conditions are

$$P(0) = 0, \qquad \left(\frac{dP}{dt}\right)_{t=0} = kv_0 \tag{1.6}$$

Differentiating (1.4) twice with respect to t, we obtain

$$\frac{d^2w}{dt^2} = \beta_0 \frac{dP}{dt} + \beta_0 \int_0^t H(t-\tau) \frac{dP}{d\tau} d\tau \tag{1.7}$$

The validity of (1.7) is implied by the folowing relations, with the conditions $y(0) = 0$ and $P(0) = 0$ taken into account

$$\frac{d}{dt} \int_0^t H(t-\tau) y(\tau) d\tau = \frac{d}{dt} \int_0^t H(\xi) y(t-\xi) d\xi = \int_0^t H(\xi) \frac{d}{dt} y(t-\xi) d\xi =$$

$$\int_0^t H(t-\tau) \frac{dy}{d\tau} d\tau = \int_0^t H(t-\tau) P(\tau) d\tau$$

$$\frac{d^2}{dt^2} \int_0^t H(t-\tau) y(\tau) d\tau = \frac{d}{dt} \int_0^t H(t-\tau) P(\tau) d\tau = \int_0^t H(t-\tau) \frac{dP}{d\tau} d\tau$$

Replacing in (1.5) d^2w / dt^2 by its expression given in (1.7), we obtain the integro-differential equation defining the unknown force. Taking into account the formulas (1.2) and (1.3) we can write this equation in the operator form

$$\frac{d^2P}{dt^2} + k\beta_t \frac{dP}{dt} + \frac{k}{m} P = 0 \tag{1.8}$$

2. In accordance with the method of integral operators [2] we write the expression for the force $P(t)$ defined by the equation (1.8) in the form

$$P(t) = v_0 kt + \Phi \{A_t^i B_t^{2n}\} \tag{2.1}$$

$$\Phi\{\cdot\} \equiv v_0 k \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{n+i}\{\cdot\} t^{2n+i+1}}{(2n+1)!\, i!}$$

Here we have the operators $A_t = A_0(1 + H^*)$ and $B_t^2 = l - A_t^2$ and the parameters $A_0 = \beta_0 k / 2$ and $l = k/m$. Expressing the operator $A_t^i B_t^{2n}$ in terms of the integral operator H^* we obtain, from (2.1),

$$P(t) = P_0(t) + \Phi\{R^*\} \tag{2.2}$$

Here

$$P_0(t) = v_0 kt + \Phi\{A_0^i B_0^{2n}\}, \quad B_0^2 = l - A_0^2, \quad \mu_0 = mk\beta_0^2 / 4$$

$$R^* = A_0^i l^n \sum_{j=1}^{i} \binom{i}{j} H^{*j} + A_0^i l^n \sum_{j=1}^{n} (-\mu_0)^j \sum_{q=1}^{2j+i} \binom{2j+i}{q} H^{*q}$$

$$H^{*r}\zeta = \int_0^t H_r(t-\tau)\zeta(\tau) d\tau \qquad (r = j, q) \tag{2.3}$$

and $H_r(t-\tau)$ are the iterated kernels of the initial lag kernel $H(t-\tau)$.

The series defining $P(t)$ converges uniformly on any finite interval of variation of t, provided that the kernel $H(t-\tau)$ is regular or weakly singular. It can be confirmed that the function $P(t)$ defined by the formula (2.1) satisfies Eq. (1.8) and the initial conditions (1.6), since

$$\frac{d^\nu}{dt^\nu} A_t^i B_t^{2n} t^{2n+i+1} = A_t^i B_t^{2n} \frac{d^\nu}{dt^\nu} t^{2n+i+1} \quad (\nu = 1, 2, \ i \geqslant 1) \tag{2.4}$$

The property of commutativity of (2.4) follows from the expression for R^* since, as it was shown in Sect. 1, the operator H^* commutes with the differential operators d^ν / dt^ν when acting on a function which vanishes, together with its first derivative at $t = 0$.

Let the operator H^* be expressed in terms of the resolvent operator $Q^*(-\eta)$ according to the formula $H^* = \chi Q^*(-\eta)$, where $\chi > 0$ and $\eta > 0$ represent the rheological constants. In this case the process of finding the iterated kernels $H_r(t-\tau)$ can be replaced by a simpler operation [2], since the power of the operator

$$H^{*r} = \chi^r \frac{(-1)^{r-1}}{(r-1)!} \frac{\partial^{r-1} Q^*(-\eta)}{\partial \eta^{r-1}} \qquad (r = 2, 3, \ldots) \tag{2.5}$$

yields, at acting on t^{2n+i+1}, the following expression

$$H^{*r} t^{2n+i+1} = \chi^r \frac{(-1)^{r-1}}{(r-1)!} \frac{\partial^{r-1}}{\partial \eta^{r-1}} \int_0^t Q(-\eta;\ t-\tau)\, \tau^{2n+i+1}\, d\tau$$

We note that in the physical sense the ratio χ / η is dimensionless and can assume any positive value.

From the expansion (2.3), for $P_0(t)$ it follows that

$$P_0(t) = \begin{cases} \dfrac{v_0 k}{B_0} e^{-A_0 t} \sin B_0 t, & mk\beta_0^2 < 4,\ B_0^2 > 0 \\ v_0 k t\, e^{-A_0 t}, & mk\beta_0^2 = 4,\ B_0^2 = 0 \\ \dfrac{v_0 k}{q_0} e^{-A_0 t} \operatorname{sh} q_0 t, & mk\beta_0^2 > 4,\ B_0^2 < 0,\ q_0^2 = A_0^2 - l > 0 \end{cases} \tag{2.6}$$

When $\chi = 0$, we have $P(t) = P_0(t)$. Therefore the function $\Phi\{R^*\}$ takes into account the influence of the hereditary factor on the force $P(t)$, while the function $P_0(t)$ corresponds to the perfectly elastic properties of the plate.

The expression for the operator $B_t^2 = B_0^2 - A_0^2(2H^* + H^{*2})$ shows that the inequality $B_t^2 < 0$ follows from the condition $B_0^2 \leqslant 0$. Therefore the condition $B_0^2 \leqslant 0$ is sufficient for the variation of the force $P(t)$ to be aperiodic. The positiveness of the operator B_t^2 is the necessary and sufficient condition for the variation of $P(t)$ to have a periodic character, while the inequality $B_0^2 > 0$ is only necessary for the realization of the periodic character since the condition $B_t^2 > 0$ implies $B_0^2 > 0$.

Finally, if $B_0^2 = 0$ and $B_t^2 = 0$ simultaneously, then $l - A_0^2 = 0$ and $l - A_t^2 = 0$, and we have $A_t^2 = A_0^2$. The latter is possible only when $H^* \equiv 0$. Since in the initial model the influence of the aftereffect is assumed, therefore the present case should be excluded from our discussion, although it does not contradict the assertion that only the condition $B_t^2 > 0$ guarantees the periodic character of the variation of the force $P(t)$ and the invariable appearance of a component of the form (2.6) when $mk\beta_0^2 < 4$ and $B_0^2 > 0$.

3. A more detailed analysis of the solution of the problem with aftereffect can only be performed by specifying the operator $Q^*(-\eta)$ and the corresponding approximation.

The operator $\vartheta_\alpha^*(-\eta)$ introduced in [3] belongs to the class of resolvent operators. Consequently, taking into account the approximation

$$\vartheta_\alpha^*(-\eta)\, t^\delta \approx \frac{q t^\delta}{1 + q\eta} \tag{3.1}$$

corresponding to the results obtained in [4] and the expression (2.5), we obtain

$$H^{*r} t^{\delta} \approx \left(\frac{\chi}{\eta}\right)^r L^r t^{\delta} \tag{3.2}$$

$$q = \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, \qquad L = \frac{q\eta}{1+q\eta}, \qquad \delta = 2n+i+1$$

$$0 < q\eta < 1, \qquad -1 < \alpha \leqslant 0, \qquad 0 < L \leqslant 1/2$$

The expression (3.2) makes it possible to express the power of the operator H^* in terms of a conventional power expression. Therefore the series (2.1) can be summed and an explicit expression for the function $P(t)$ obtained in the closed form.

In the case $B_t^2 > 0$ which is of interest in the study of the process of printing, we have

$$P(t) \approx \frac{v_0 k}{B_1} e^{-A_1 t} \sin B_1 t \tag{3.3}$$

$$A_1(t) = A_0\left(1 + \frac{\chi}{\eta} L\right), \qquad B_1^2(t) = \frac{k}{m} - A_1^2(t)$$

Here $A_1 > A_0$ while $B_1^2 < B_0^2$ and we also have

$$\frac{k}{m}\left[1 - \mu\left(1 + \frac{\chi}{2\eta}\right)^2\right] < B_1^2 < \frac{k}{m}(1-\mu) \tag{3.4}$$

since $0 < L \leqslant 1/2$ and $\mu = mk\beta_1^2/4 < 1$.

The duration of impact can be found from the equation

$$tB_1(t) = \pi, \qquad \text{if} \quad mA_1^2(t) < k \tag{3.5}$$

The stamp rebounds from the viscoelastic plate at the instant $t = t_*$, where t_* is the root of (3.5) and $t_* > t_1$, where $t_1 = \pi / B_0$ being the instant of rebound of the stamp from a perfectly elastic plate. It follows that the duration of the impact is greater in the presence of a lag, than without it. The inequalities (3.4) imply that the first condition of (2.6) guaranteeing the rebound of the stamp from the perfectly elastic plate, cannot guarantee that the stamp rebounds from a viscoelastic plate.

We note that here the rheological properties of the plate play a decisive role, since the variation in the value of the ratio χ/η appearing in the formulas (3.3) may be very large (naturally with the condition $B_1^2 > 0$ observed), while the fractional dimensionless multiplier L is bounded by the inequalities $0 < L < 1/2$.

Since formally $0 < \chi/\eta < \infty$, we have not excluded unsuccessful case in which it is practically impossible to bring the ratio k/m to the value at which a rebound of the stamp is guaranteed, i.e. to attain the condition $B_1^2 > 0$, unless the value of L can be reduced in the corresponding manner. The value of the recovery coefficient $k^{-1}(dP/dt)_{t=t_*}$ is smaller when the aftereffect is present, than when it is absent.

Finally, the influence of the aftereffect is reflected in the increased displacement of the plate, and the ratio $\chi/2\eta$ determines the relative increase in its maximum value. If for some reason it is required that the stamp does not rebound from the viscoelastic plate with a spring present between them, then we must proceed from the solution corresponding to the condition $B_0^2 < 0$. Then the force $P(t)$ will vary aperiodically and, in accordance with the procedure derived above, its approximate expression can then be written in the form

$$P(t) \approx \frac{v_0 k}{q_1} e^{-A_1 t} \operatorname{sh} q_1 t, \qquad q_1(t) = A_1^2 - l > 0 \tag{3.6}$$

The curve (3.6) has a single maximum and a horizontal asymptotics coinciding with the t-axis.

The influence of the elastic lag leads to reduction in the value of P_{max} and in the ordinate of the point of inflection and, compared with the perfectly elastic case, they are both displaced to the right. In the perfectly elastic case $P(t)$ decreases with $t \to \infty$ at a slower rate than in the case when elastic lag is present.

The case of $B_0^2 = 0$ and $B_t^2 = 0$ also corresponds to the aperiodic variation of the force $P(t) > 0$ and, when $t \to \infty$, we have $\lim P(t) = 0$. Its expression $P(t) = P_0(t)$ is given by the formula (2.5). Looking at this case from the elastic lag angle we find, that we can regard it as the neutral case, since it must be encountered when the law of variation of $P(t)$ changes from aperiodic to periodic, the change taking place when the conditions ensuring the existence of the inequality $B_0^2 < 0$ are violated. We can eliminate such a situation by increasing the mass m of stamp to such an extent that Eq. (1.8) can be replaced by its limiting form obtained when $m \to \infty$, i.e.

$$\frac{d^2P}{dt^2} + k\beta_t \frac{dP}{dt} = 0 \tag{3.7}$$

The solution of (3.7) with the initial conditions (1.6) has the form

$$P(t) = \frac{1}{\beta_0}(1 - e^{-\beta_0 kt}) - k \sum_{n=2}^{\infty} \frac{(-\beta_0 k)^{n-1}}{n!} R_{n-1}$$

$$R_{n-1} = \sum_{i=1}^{n-1} \binom{n-1}{i} \frac{(-\chi)^{i-1}}{(i-1)!} \frac{\partial^{i-1}}{\partial \eta^{i-1}} \int_0^t \partial_\alpha(-\eta;\ t-\tau)\tau^n \, d\tau \tag{3.8}$$

where $\partial_\alpha(-\eta;\ t-\tau)$ is the lag kernel due to Rabotnov [4]. When $\chi = 0$, the curve (3.8) has a horizontal asymptotics separated from the t-axis by the distance of $1/\beta_0$. The asymptotics shifts downwards when $\chi \neq 0$.

The result obtained can be studied in greater detail using the approximation (3.1). Then we have

$$P(t) \approx \frac{1}{\beta_1}(1 - e^{-\beta_1 kt})$$

$$\beta_1(t) = \beta_1\left(1 + \frac{\chi t^{1+\alpha}}{\Gamma(2-\alpha) + \eta t^{1+\alpha}}\right) \tag{3.9}$$

Expression (3.9) implies that the position of the horizontal asymptotics of the curve (3.8) is determined by the quantity $(\eta/\beta_0)(\chi + \eta)^{-1}$ and the degree of its descent in comparison with the perfectly elastic case depends on the magnitude of the ratio χ/η. Thus, when $\chi = \eta$, the elastic lag produces a downward shift of the asymptotics which is twice as large as the shift occurring in the case of an elastic plate.

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